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ON MAXIMIZING THE SPEED OF A RANDOM WALK IN FIXED ENVIRONMENTS

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Abstract. We consider a random walk in a fixed $\mathbb{Z}$ environment composed of two point types: $(q, 1-q)$ and $(p, 1-p)$ for $\frac{1}{2} < q < p$. We study the expected hitting time at N for a given number k of p-drifts in the interval $[1, N-1]$, and find that this time is minimized asymptotically by equally spaced p-drifts.

1. Introduction

Procaccia and Rosenthal [1] studied how to optimally place given number of vertices with a positive drift on top of a simple random walk to minimize the expected crossing time of an interval. They ask about extending their work to the situation where the environment on $\mathbb{Z}$ is composed of two point types: $(q, 1-q)$ and $(p, 1-p)$ for $\frac{1}{2} < q < p$. This is the goal of this note. See [1] for background and further related work.

Consider nearest neighbor random walks on $0, 1, ..., N$ with reflection at the origin. We denote the random walk by $\{X_n\}_{n=0}^{\infty}$, and by $\omega(i)$ the transition probability at vertex i:

$$\begin{aligned} P(X_{n+1} = i+1 | X_n = i) &= \omega(i) \\ P(X_{n+1} = i-1 | X_n = i) &= 1 - \omega(i). \end{aligned}$$

First, we prove the following proposition concerning the expected hitting time at vertex N:

Proposition 1. *For a walk ω starting at x, the hitting time $T_N = \min\{n \geq 0 | X_n = N\}$ satisfies:*

$$E_{\omega}^{x}(T_N) = N - x + 2\sum_{i=x}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k,$$

where $\rho_i = \frac{1-\omega(i)}{\omega(i)}$, and $E_{\omega}^{x}(T_N)$ stands for the expected hitting time. In particular:

$$E_{\omega}^{0}(T_N) = N + 2\sum_{i=1}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k.$$

Corollary 2. *The expected hitting time from 0 to N is symmetric under reflection of the environment, i.e. taking the environment $\omega'(i) = \omega(N-i)$ gives $E_{\omega'}^{0}(T_N) = E_{\omega}^{0}(T_N)$.*

Next we turn to the case of an environment consisting of two types of drifts, $(q,1-q)$ (i.e. probability q to go to the right and $1-q$ to the left) and $(p,1-p)$, for some $\frac{1}{2}<q<p\leq 1$. Assume that k of the vertices are p-drifts, and the rest are q-drifts. In [1] it was proven that for $q=\frac{1}{2}$ equally spaced p-drifts minimize $\frac{E_{\omega}^{0}(T_N)}{N}$ (for large N). In this paper we extend this result for $q>\frac{1}{2}$. We define an environment in which the p-drifts are equally spaced (up to integer effects):

$$\omega_{N,k}\left(x\right)=\begin{cases} p & x=\left\lfloor i\cdot\frac{N-1}{k}\right\rfloor \text{ for some } 1\leq i\leq k\\ q & \text{otherwise}\end{cases},$$

and prove the following theorem:

Theorem 3. *For every $\varepsilon>0$ there exists n_0 such that for every $N>n_0$ and environment ω:*

$$\frac{E_{\omega}^{0}\left(T_N\right)}{N}>\frac{E_{\omega_{N,k}}^{0}\left(T_N\right)}{N}-\varepsilon,$$

where k is the number of p-drifts in ω.

Finally, we consider the set of environments $\omega_{ak,k}$ for some $a\in\mathbb{N}$, and calculate $\lim\limits_{k\to\infty}\frac{E_{\omega_{ak,k}}^{0}(T_N)}{ak}$:

Proposition 4. *Let $a\in\mathbb{N}$. Then:*

$$\lim_{k\to\infty}\frac{E_{\omega_{ak,k}}^{0}\left(T_{ak}\right)}{ak}=1+\frac{2}{a}\cdot\frac{\alpha^{a+2}-a\alpha^{3}+\left(a-1\right)\alpha^{2}+\left(\left(a\alpha^{2}-\left(a+1\right)\alpha\right)\alpha^{a}+\alpha\right)\beta}{\left(\alpha^{2}-2\alpha+1\right)\alpha^{a}\beta-\alpha^{3}+2\alpha^{2}-\alpha}.$$

2. Proof of the main theorem

Proof of Proposition 1 . Define $v_x=E_{\omega}^{x}\left(T_N\right)$ for $0\leq x\leq N$. By conditioning on the first step:

(1) $v_N=0$
(2) $v_0=v_1+1$
(3) $v_x=p_x v_{x+1}+\left(1-p_x\right)v_{x-1}+1 \qquad 1\leq x\leq N-1.$

To solve these equations, define $a_x=v_x-v_{x-1}$ (for $1\leq x\leq N$) and $b_x=v_{x+1}-v_{x-1}$ (for $1\leq x\leq N-1$). Then:

$$\begin{aligned} b_x &= a_x+a_{x+1}\\ a_x &= p_x b_x+1\\ a_1 &= -1\end{aligned}$$

We get for a_x the relation $a_{x+1}=\rho_x a_x-\rho_x-1$, whose solution is $a_x=-2\sum\limits_{j=1}^{x-1}\prod\limits_{k=j}^{x-1}\rho_k-1$, and then:

$$
\begin{aligned}
v_x &= \sum_{i=x+1}^{N} (v_{i-1} - v_i) + v_N \\
&= \sum_{i=x+1}^{N} (-a_i) + v_N \\
&= N - x + 2 \sum_{i=x}^{N-1} \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k
\end{aligned}
$$

□

Definition 5. To evaluate $E_\omega^0 (T_N)$ we define:

$$
S_N = \sum_{i=1}^{N-1} \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k = \sum_{d=1}^{N-1} \sum_{j=1}^{N-d} \prod_{k=j}^{j+d-1} \rho_k .
$$

Next define $\widetilde{\rho}_k$ for k in the circle $\mathbb{Z}_{N-1}$, such that for $1 \leq k \leq N-1$ we will have $\widetilde{\rho}_k = \rho_k$ (gluing the point 0 to the point $N-1$), and then look at:

$$
\widetilde{S}_N = \sum_{d=1}^{N-1} \sum_{j=1}^{N-1} \prod_{k=j}^{j+d-1} \widetilde{\rho}_k .
$$

This way, rather than summing $\prod_{k=i}^{j} \rho_k$ over subintervals $[i,j]$ of $[1, N-1]$, we sum $\prod_{k=i}^{j} \widetilde{\rho}_k$ over subintervals of the circle $\mathbb{Z}_{N-1}$.

Proposition 6. *Define* $\alpha = \frac{1-q}{q}$, $\beta = \frac{1-p}{p}$. *Since* $\beta < \alpha < 1$*:*

$$
\begin{aligned}
\left| \widetilde{S}_N - S_N \right| &= \sum_{d=1}^{N-1} \sum_{j=N-d+1}^{N-1} \prod_{k=j}^{j+d-1} \rho_k \\
&\leq \sum_{d=1}^{N-1} d\alpha^d \\
&\leq \sum_{d=1}^{\infty} d\alpha^d < C(\alpha)
\end{aligned}
$$

for some constant $C(\alpha)$ which doesn't depend on N.

Definition 7. Let $n_i^{(d)}$ be the number of p-drifts in the interval $[i, i+d-1]$.

Since every drift appears in d intervals of length d, $\sum_{i=1}^{N-1} n_i^{(d)} = dk$. Also,

$$\begin{aligned}\widetilde{S}_N &= \sum_{d=1}^{N-1}\sum_{i=1}^{N-1}\left(\frac{\beta}{\alpha}\right)^{n_i^{(d)}}\cdot\alpha^d \\ &= \sum_{d=1}^{N-1}\sigma_d\end{aligned}$$

where $\sigma_d = \sum\limits_{i=1}^{N-1}\left(\frac{\beta}{\alpha}\right)^{n_i^{(d)}}\cdot\alpha^d$.

Claim 8. For $n_l^{(d)}\in\mathbb{N}$ the expression σ_d is minimized under the restriction $\sum\limits_{l=1}^{N-1} n_l^{(d)} = dk$ if $n_i^{(d)} - n_j^{(d)} \le 1$ for all i,j.

Proof. For convenience, we omit d from the notation, and set $\mathbf{n}=(n_1,...,n_{N-1})$. If a vector $\mathbf{n}$ satisfies $n_i - n_j \le 1\,\forall i,j$, we say $\mathbf{n}$ is almost constant. We will show that σ is minimal for some almost constant vector. Then we show that σ takes on the same value for all almost constant vectors under the restriction, and this completes the proof.

Suppose σ is minimized (under the restriction) by some vector $\mathbf{n}^0$. If $\mathbf{n}^0$ is almost constant, we are done. Else, for some i,j we have that $n_i^0 - n_j^0 \ge 2$. We choose i,j such that $n_i^0 - n_j^0$ is maximal. Define:

$$n_l^1 = \begin{cases} n_l^0 & l\ne i,j \\ n_l^0 - 1 & l = i \\ n_l^0 + 1 & l = j\end{cases}.$$

$\mathbf{n}^1$ satisfies the restriction, and $\sigma\left(\mathbf{n}^0\right)\ge\sigma\left(\mathbf{n}^1\right)$:

$$\begin{aligned}\sigma\left(\mathbf{n}^0\right)-\sigma\left(\mathbf{n}^1\right) &= \sum_{t=1}^{N-1}\left(\frac{\beta}{\alpha}\right)^{n_t^0}\cdot\alpha^d - \sum_{t=1}^{N-1}\left(\frac{\beta}{\alpha}\right)^{n_t^1}\cdot\alpha^d \\ &= \alpha^d\left(\left(\frac{\beta}{\alpha}\right)^{n_i^0} + \left(\frac{\beta}{\alpha}\right)^{n_j^0} - \left(\frac{\beta}{\alpha}\right)^{n_i^0-1} - \left(\frac{\beta}{\alpha}\right)^{n_j^0+1}\right) \\ &= \alpha^d\left(1-\frac{\beta}{\alpha}\right)\left(\left(\frac{\beta}{\alpha}\right)^{n_j^0} - \left(\frac{\beta}{\alpha}\right)^{n_i^0-1}\right) \\ &\ge 0,\end{aligned}$$

where the inequality follows from the fact that $0\le\frac{\beta}{\alpha}<1$ and $n_j^0 < n_i^0 - 1$. From minimality of $\sigma\left(\mathbf{n}^0\right)$, we get that $\sigma\left(\mathbf{n}^1\right)$ is also minimal. This process must end after a finite number of steps f, yielding an almost constant $\mathbf{n}^f$ which minimizes σ.

Now for a general almost constant vector $\mathbf{n}$, set $a=\min\{n_l : 1\le l\le N-1\}$. We have $n_l\in\{a,a+1\}$, so defining m_0 to be the number of a's and m_1 to be the number of $a+1$'s, we get:

$$\begin{aligned} dk &= \sum_{l=1}^{N-1} n_l \\ &= m_0 a + m_1 (a+1) \\ &= (m_0 + m_1) a + m_1 \\ &= (N-1) a + m_1, \end{aligned}$$

and since $m_1 < N-1$, there is a unique solution for natural a, m_1. So all almost constant $\mathbf{n}$ (satisfying the restriction) are the same up to ordering, and since σ doesn't depend on the order, they all give the same value. □

Claim 9. For every choice of M, k, the placement of k drifts on the circle $\mathbb{Z}_M$ in which the ith drift is at the point $\left\lfloor i \cdot \frac{M}{k} \right\rfloor$ satisfies:

$$\forall d, i, j \quad n_i^{(d)} - n_j^{(d)} \leq 1.$$

Proof. Place the ith drift at the point $\left\lfloor i \cdot \frac{M}{k} \right\rfloor$. We calculate the number of drifts in the interval $[x, x+d-1]$. The first drift inside this interval is:

$$\begin{aligned} \left\lfloor i_0 \cdot \frac{M}{k} \right\rfloor &\geq x \\ i_0 \cdot \frac{M}{k} &\geq x \\ i_0 &\geq x \cdot \frac{k}{M} \\ i_0 &= \left\lceil x \cdot \frac{k}{M} \right\rceil. \end{aligned}$$

The last drift inside this interval is:

$$\begin{aligned} \left\lfloor i_1 \cdot \frac{M}{k} \right\rfloor &\leq x + d - 1 \\ i_1 \cdot \frac{M}{k} &< x + d \\ i_1 &< (x+d) \cdot \frac{k}{M} \\ i_1 &= \left\lceil (x+d) \cdot \frac{k}{M} \right\rceil - 1. \end{aligned}$$

The number of drifts inside this interval is therefore:

$$
\begin{aligned}
i_1 - i_0 + 1 &= \left\lceil (x+d) \cdot \frac{k}{M} \right\rceil - \left\lceil x \cdot \frac{k}{M} \right\rceil \\
&\geq (x+d) \cdot \frac{k}{M} - x \cdot \frac{k}{M} - 1 \\
&= \frac{dk}{M} - 1 \\
i_1 - i_0 + 1 &\leq (x+d) \cdot \frac{k}{M} + 1 - x \cdot \frac{k}{M} \\
&= \frac{dk}{M} + 1.
\end{aligned}
$$

So for non-integer $\frac{dk}{M}$ the number of drifts takes on only the two values $\left\lfloor \frac{dk}{M} \right\rfloor, \left\lceil \frac{dk}{M} \right\rceil$. For integer $\frac{dk}{M}$ we simply have:

$$
\begin{aligned}
i_1 - i_0 + 1 &= \left\lceil (x+d) \cdot \frac{k}{M} \right\rceil - \left\lceil x \cdot \frac{k}{M} \right\rceil \\
&= \frac{dk}{M}
\end{aligned}
$$

□

Claim 10. $\widetilde{S}_N$ is minimal for the configuration of drifts described by $\omega_{N,k}$ (where the ith drift is at vertex $\left\lfloor i \cdot \frac{N-1}{k} \right\rfloor$).

Proof. $\widetilde{S}_N = \sum\limits_{d=1}^{N-1} \sigma_d$, and by claims 8 and 9 each σ_d is minimized by this configuration, therefore the sum is also minimized. □

Proof of Theorem 3. From Proposition 6, $0 < \widetilde{S}_N - S_N < C$. Let $n_0 = \frac{2C}{\varepsilon}$. Then for $N > n_0$:

$$
\begin{aligned}
\frac{E^0_\omega (T_N)}{N} &= \frac{N + 2S_N}{N} \\
&= 1 + 2\frac{S_N}{N} \\
&> 1 + 2\frac{\widetilde{S}_N}{N} - \varepsilon \\
&\geq 1 + 2\frac{\widetilde{S}^*_N}{N} - \varepsilon \\
&\geq 1 + 2\frac{S^*_N}{N} - \varepsilon \\
&= \frac{E^0_{\omega_{N,k}} (T_N)}{N} - \varepsilon
\end{aligned}
$$

where we denote by S^*_N and $\widetilde{S}^*_N$ the values caculated for $\omega_{N,k}$. □

Proof of Proposition 4. We evaluate $\lim\limits_{k\to\infty} \frac{\widetilde{S}^*_{ak}}{ak}$. First, we consider the k intervals that do not contain any β, each of which contributes:

$$s_0 = \sum_{i=1}^{a-1} (a-i)\,\alpha^i.$$

Next we consider the k intervals that contain $n \geq 1$ β's:

$$s_n = \beta^n \cdot \alpha^{(a-1)(n-1)} \cdot \sum_{r=0}^{a-1} \sum_{s=0}^{a-1} \alpha^{r+s}.$$

Then we get:

$$\begin{aligned}
\lim_{k\to\infty} \frac{\widetilde{S}^*_{ak}}{ak} &= \frac{1}{a} \lim_{k\to\infty} \frac{ks_0 + \sum\limits_{n=1}^{k} ks_n}{k} \\
&= \frac{1}{a} \cdot \frac{\alpha^{a+2} - a\alpha^3 + (a-1)\,\alpha^2 + \left(\left(a\alpha^2 - (a+1)\,\alpha\right)\alpha^a + \alpha\right)\beta}{\left(\alpha^2 - 2\alpha + 1\right)\alpha^a\beta - \alpha^3 + 2\alpha^2 - \alpha},
\end{aligned}$$

and since $\lim\limits_{k\to\infty} \frac{\widetilde{S}^*_{ak} - S^*_{ak}}{ak} = 0$ from Proposition 6, the proof is complete. □

3. Further questions

(1) Show that the optimal environment also minimizes the variance of the hitting time.
(2) Can this result be extended to a random walk on $\mathbb{Z}$ with a given density of drifts (as in [1])?
(3) Can similar results be found for other graphs? For example, $\mathbb{Z}_2 \times \mathbb{Z}_N$.

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References

1. E.B. Procaccia and R. Rosenthal, *The need for speed: maximizing the speed of random walk in fixed environments*, Electronic Journal of Probability **17** (2012), 1–19.